

UNENTANGLED MEASUREMENTS AND FRAME FUNCTIONS

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ABSTRACT. Gleason's theorem shows the equivalence of von Neumann's density operator formalism of quantum mechanics and frame functions that arise from associating measurement probabilities to vectors on a sphere. Wallach's *Unentangled Gleason's Theorem* extends it to the regime of measurements by unentangled orthogonal bases (UOB) in a multi-partite system in which each subsystem is at least 3-dimensional. In this paper, we determine the structure of unentangled frame functions in general. We first classify the unentangled frame functions for multi-qubit systems, and then extend it to factors of varying dimensions including countably infinite dimensions (separable Hilbert spaces). The unentangled frame functions have a remarkable combinatorial structure suggesting possible fundamental interpretations.

1. INTRODUCTION

In von Neumann's [1] approach to quantum mechanics, the formalism assumes a factor algebra with an endowed trace function. The mixed states in this set up are the self-adjoint, positive elements of trace 1 looked upon as defining a measure on the set of self-adjoint, idempotent elements. This led Mackey to ask the natural question: Is every such measure given by a mixed state? Gleason gave an affirmative answer to this question in the case of factors of type I_n with $n > 2$ (i.e. the bounded operators on a separable Hilbert space). In his approach to his proof he introduced the notion of frame function (a non-negative function on the pure states that sums to 1 on an orthonormal basis of the Hilbert space) which is easily seen to be equivalent to that of measure on the set of self-adjoint idempotents. Gleason's theorem shows that such a function, f , must be of the form $f(v) = \langle v|A|v\rangle$ for v any unit norm vector in the Hilbert space and A a non-negative, self-adjoint operator of trace 1 if the dimension of the Hilbert space is at least 3. In many contexts of quantum information theory that deal with state spaces of many independent particles, the only pure states that are sampled are product (or unentangled) states. This led the last named author [3] to ask if such a sampling of only unentangled states would allow for more general frame functions. In this paper we classify all the unentangled frame functions, for an arbitrary but finite number of tensor factors, including those of countably infinite dimension (separable Hilbert spaces), completing the treatment of the unentangled frame functions.

The organization of the rest of the paper is as follows. We first introduce the idea of an *unentangled frame function* in Section 2. In Section 3, we choose the fundamental domain in $\mathbb{P}^1(\mathbb{C})$ that we use to identify a vector with its orthogonal. In Section 4 we classify all multi-qubit frame functions. Section 5 generalizes this classification further to include the case when some of the tensor factors are of dimensions at least 3 and at most countable (separable Hilbert spaces). Section 6 concludes our discussion with some remarks.

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2. UNENTANGLED FRAME FUNCTIONS

Let us take a brief look at unentangled Gleason setup as in [3], and make more precise the ideas from the Introduction. Let

$$\mathcal{H} = H_1 \otimes H_2 \otimes \cdots \otimes H_n$$

with $\dim H_i \geq 2$. Technically we should be looking at the completed tensor product if two or more of the factors are infinite dimensional. However, since we will only be looking at product vectors this will not be necessary. Applying a permutation of the factors we may assume that the first k of the H_i are of dimension 2 and all of the rest have dimension > 2 . Thus

$$\mathcal{H} = \otimes^k \mathbb{C}^2 \otimes H_{k+1} \otimes \cdots \otimes H_n$$

with $\dim H_i > 2$ and if $n = k$ then by convention the last factor is \mathbb{C} .

It is given the tensor product Hilbert structure, $\langle \dots | \dots \rangle$. A vector in \mathcal{H} is called unentangled (a product vector) if it is a tensor product of unit vectors, one from each H_i factor. Two such vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ and $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ are orthogonal

$$\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n | w_1 \otimes w_2 \otimes \cdots \otimes w_n \rangle = 0$$

if and only if there is at least one i with $\langle v_i | w_i \rangle = 0$. A UOB $\{u_i\}$ is a basis of \mathcal{H} consisting of orthogonal (unit norm) unentangled vectors. Let Σ be the set of all unentangled vectors in \mathcal{H} .

Definition 2.1. An unentangled frame function is a map

$$f: \Sigma \rightarrow \mathbb{R}_{\geq 0}, \tag{1}$$

such that for every UOB $\{u_i\}$,

$$\sum_i f(u_i) = 1. \tag{2}$$

In the next few sections we will be dealing with the multi-qubit case, so we specialize the above to prepare for it. Let $\mathcal{H}_n = \otimes^n \mathbb{C}^2$ be the space of n qubits. A UOB is then a basis $\{u_0, u_1, \dots, u_{2^n-1}\}$ of \mathcal{H}_n consisting of orthogonal (unit norm) unentangled vectors.

3. A FUNDAMENTAL DOMAIN

Let $\sigma: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be defined by $(x, y) \mapsto (-\bar{y}, \bar{x})$. We note that $\langle x | \sigma x \rangle = 0$ and if x is a state then up to phase σx is the unique state perpendicular to x . The σ induces a map of $\mathbb{P}^1(\mathbb{C})$ to itself which we also denote by σ . We have the standard map

$$\mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{C} \cup \infty$$

given by

$$(x, y) \mapsto \frac{x}{y}.$$

We note that under this identification as a map from $\mathbb{C} \cup \infty$ to itself

$$\sigma z = -\frac{1}{\bar{z}}.$$

In particular, on S^1 it is given by $z \mapsto -z$.

Here is a simple fundamental domain for σ :

$$F = \{z \in \mathbb{C} \mid |z| < 1\} \cup \{z \in \mathbb{C} \mid |z| = 1, \operatorname{Im} z > 0\} \cup \{0\}.$$

A frame function f in a single qubit space corresponds to an arbitrary function $\phi: F \rightarrow [0, 1]$ by letting $f(a) = \phi(a)$ if $a \in F$ and $f(a) = \phi(\sigma a)$ if $a \notin F$.

4. UNENTANGLED FRAME FUNCTIONS IN n QUBITS

Let \mathcal{H}_n denote n -qubit space. We choose a fundamental domain F for the map $[z] \rightarrow \sigma([z]) = [\hat{z}]$ in one qubit. We set $F_n = F \otimes F \otimes \cdots \otimes F$ (n copies). Let $\Omega_n = \{1, \dots, n\}$. If $J \subset \Omega_n$ is a subset we define $\sigma_J = T_1 \otimes T_2 \otimes \cdots \otimes T_n$ with $T_j = \sigma$ if $j \in J$ and $T_j = Id$, if $j \notin J$. We set $J^c = \Omega_n - J$. We note

$$\Sigma = \cup_{J \subset \Omega_n} \sigma_J(F_n)$$

is a disjoint union. If $z = z_1 \otimes \cdots \otimes z_n \in F_n$, if $J \subset \Omega_n$ and if $J^c = \{i_1, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$ then we set $\tau_J(z) = (z_{i_1}, \dots, z_{i_k})$ if $J \neq \Omega_n$, if $J = \Omega_n$ then use the symbol ω for the value.

Lemma 4.1. *If f is a function on the product states of \mathcal{H}_n such that f sums to a fixed constant c on all UOB's then for each $J \subset \Omega_n$ there exists a function ϕ_J on $F^{n-|J|}$ (direct product of $n - |J|$ copies of F , $F^0 = \{\omega\}$, $\phi_{\Omega_n}(\omega) = c$) such that if $z \in F_n$ then*

$$\sum_{L \subset J} f(\sigma_L(z)) = \phi_J(\tau_J(z)).$$

Proof. After permuting the factors we may assume that $J = \{1, \dots, j\}$, thus the sum on the left hand side is

$$\sum_{I \subset \{1, \dots, j\}} f(\sigma_I(z_1 \otimes \cdots \otimes z_j) \otimes z_{j+1} \otimes \cdots \otimes z_n)$$

call it b . Observing that if the set

$$Z = \{(\sigma_I(z_1 \otimes \cdots \otimes z_j) \otimes z_{j+1} \otimes \cdots \otimes z_n) | I \subset J\}$$

is extended to a UOB by adjoining elements u_1, \dots, u_r with $r = 2^j(2^{n-j} - 1)$ then $\sum f(u_i) = 1 - b$ no matter how we found the extension. Also Z is an orthonormal basis of $\otimes^j \mathbb{C}^2 \otimes z_{j+1} \otimes \cdots \otimes z_n$ which only depends on $z_{j+1} \otimes \cdots \otimes z_n$. This proves the result. \square

Now applying inclusion exclusion we have in the notation of the previous lemma

Lemma 4.2. *If $z \in F_n$ and if f is a function on the product states satisfying the hypotheses of Lemma 4.1 then*

$$f(\sigma_J(z)) = \sum_{L \subset J} (-1)^{|J-L|} \phi_L(\tau_L(z)).$$

Proof. Inclusion exclusion says: Let α and β be functions from the set of all subsets of Ω_n to \mathbb{C} and such that if $J \subset \Omega_n$ then

$$\sum_{L \subset J} \alpha(L) = \beta(J),$$

then

$$\alpha(J) = \sum_{L \subset J} (-1)^{|J-L|} \beta(L).$$

(c.f. Rota [5]). The lemma follows from this assertion by taking $\alpha(J) = f(\sigma_J(z))$ and $\beta(J) = \phi_J(\tau_J(z))$. \square

Theorem 4.3. *If $L \subsetneq \Omega_n$ let ϕ_L be a real valued function on $F_{n-|L|}$. Assume that $\phi_{\Omega_n}(\omega) = c$ then the function $f : \mathcal{H}_n \rightarrow \mathbb{R}$ defined by*

$$f(\sigma_J(z)) = \sum_{L \subset J} (-1)^{|J-L|} \phi_L(\tau_L(z))$$

for $J \subset \Omega_n$ and $z \in F_n$ satisfies

$$\sum_{i=1}^{2^n} f(z_i) = c.$$

if z_1, \dots, z_{2^n} is a UOB.

Proof. We first note that if $z \in F_{n-1}$ and $J \subset \{2, \dots, n\}$ then

1. $f(a \otimes \sigma_J(z)) + f(\hat{a} \otimes \sigma_J(z))$ is independent of a .

Indeed,

$$f(a \otimes \sigma_J(z)) = \sum_{L \subset J} (-1)^{|J|-|L|} \phi_L(\tau_L(a \otimes z))$$

and (if $1 \in L$ write $L = \{1\} \cup L'$ with $L' \subset \{2, \dots, n\}$)

$$\begin{aligned} f(\hat{a} \otimes \sigma_J(z)) &= f(\sigma_{J \cup \{1\}}(a \otimes z)) = \sum_{L \subset J \cup \{1\}} (-1)^{|J|+1-|L|} \phi_L(\tau_L(a \otimes z)) = \\ &= - \sum_{L \subset J} (-1)^{|J|-|L|} \phi_L(\tau_L(a \otimes z)) + \sum_{L' \subset J} (-1)^{|J|-|L'|} \phi_{L' \cup \{1\}}(\tau_{L' \cup \{1\}}(a \otimes z)). \end{aligned}$$

We therefore have

$$f(a \otimes \sigma_J(z)) + f(\hat{a} \otimes \sigma_J(z)) = \sum_{L' \subset J} (-1)^{|J|-|L'|} \phi_{L' \cup \{1\}}(\tau_{L' \cup \{1\}}(a \otimes z)).$$

This proves 1. since $\tau_{L' \cup \{1\}}(a \otimes z)$ is independent of a .

We will now prove the theorem by induction on n . If $n = 1$ the assertion is that if $a \in F$ then $f(a) + c - f(a) = c$. So the result is true for $n = 1$. Now assume the result for $n - 1 \geq 1$. We now prove it for n . If $a \in F$ then define $f_a(z) = f(a \otimes z)$. Then if $J \subsetneq \Omega = \{2, \dots, n\}$ and $z \in F_{n-1}$ we have

$$f_a(\sigma_J(z)) = \sum_{L \subset J} (-1)^{|J-L|} \phi_L(\tau_L(a \otimes z))$$

and

$$f_a(\sigma_{\Omega}(z)) = \phi_{\Omega}(a).$$

Thus the inductive hypothesis implies that f_a satisfies

$$\sum_{i=1}^{2^{n-1}} f_a(z_i) = \phi_{\Omega}(a).$$

for any UOB $\{z_1, \dots, z_{2^{n-1}}\}$ of \mathcal{H}_{n-1} .

We are now ready to prove the theorem. Let $\mathcal{B} = \{z_1, \dots, z_{2^n}\}$ be a UOB. Then Theorem 6 in [3] implies that there exist $a_1, \dots, a_r \in F$, V_1, \dots, V_r orthogonal subspaces of \mathcal{H}_{n-1} such that

$$\mathcal{H}_{n-1} = V_1 \oplus \dots \oplus V_r$$

and $u_{ij}, j = 1, \dots, d_i$ and $v_{ij}, j = 1, \dots, d_i$ orthonormal basis of V_i consisting of product vectors such that

$$\mathcal{B} = \{a_i \otimes u_{ij} | i = 1, \dots, r, j = 1, \dots, d_i\} \cup \{\hat{a}_i \otimes v_{ij} | i = 1, \dots, r, j = 1, \dots, d_i\}.$$

For each i we apply the inductive hypothesis to f_{a_i} and find that

$$\sum_{j=1}^{d_i} f_{a_i}(u_{i,j})$$

depends only on a_i and V_i and not on the particular orthonormal basis of V_i . Thus we can replace $u_{i,j}$ with $v_{i,j}$ without changing the sum. Now

$$f(a_i \otimes v_{i,j}) + f(\hat{a}_i \otimes v_{i,j})$$

is independent of a_i by 1. Thus we can replace all of the a_i with a fixed element $a \in F$ without changing the sum. Thus the sum is given by

$$\sum_{ij} f(a \otimes v_{ij}) + \sum_{ij} f(\hat{a} \otimes v_{ij}).$$

We now observe that if we define

$$g(z) = f(a \otimes z) + f(\hat{a} \otimes z)$$

then (see the proof of 1.)

$$g(\sigma_J(z)) = \sum_{L' \subset J} (-1)^{|J|-|L'|} \phi_{L' \cup \{1\}}(\tau_{L' \cup \{1\}}(a \otimes z)).$$

and

$$\sum_{J \subset \Omega} g(\sigma_J(z)) = c$$

for all $z \in F_{n-1}$. Finally we can apply the inductive hypothesis to replace the basis $\{v_{ij}\}$ by $\{\sigma_J(z) | J \subset \Omega\}$ with $z \in F_{n-1}$ and the theorem is proved. \square

Corollary 4.4. *If f is a function on the product state such that there exists a constant c and for every generic UOB, $\{z_1, \dots, z_{2^n}\}$ we have*

$$\sum f(z_i) = c$$

then the same is true for every UOB.

Proof. The characterization of the generic UOB in [4] makes it clear that if f sums to c on all generic UOB then f and the functions ϕ_J have the property in Lemma 4.1. Lemma 4.2 and Theorem 4.3 now complete the proof. \square

Notice that the corollary is interesting only when $n \geq 3$, since when $n < 3$, every UOB is part of a maximal dimensional family.

5. GENERAL UNENTANGLED FRAME FUNCTIONS

In this section we will give a complete description of unentangled frame functions for separable Hilbert spaces of the form

$$\mathcal{H} = H_1 \otimes H_2 \otimes \cdots \otimes H_n$$

with $\dim H_i \geq 2$.

Let f be an unentangled frame function on \mathcal{H} . Thus f is a real valued function product states such that there exists a scalar c such that if $\{u_i\}$ is a UOB then $\sum f(u_i) = c$.

If z is a product state in $\otimes^k \mathbb{C}^2$ then setting $f_z(x) = f(z \otimes x)$ for x a product state in $H_{k+1} \otimes \cdots \otimes H_r$, f_z is an unentangled frame function on $H_{k+1} \otimes \cdots \otimes H_r$ with

$$\sum f_z(u_i) = c(z)$$

for $\{u_i\}$ a UOB of $H_{k+1} \otimes \cdots \otimes H_r$ (so $c(z) = \sum_i f(z \otimes u_i)$ depends only on z and f). The unentangled Gleason theorem [3] implies that there exists $A(z)$ a trace class, self-adjoint non-negative operator on $H_{k+1} \otimes \cdots \otimes H_r$ such that

1. $\text{tr} A(z) = c(z)$.
2. $f_z(u) = \langle u | A(z) | u \rangle$.

Note that the proof of the unentangled Gleason theorem given in [3] does not use finite dimensionality and therefore applies to the context of this paper.

This proves the following reduction of the problem.

Lemma 5.1. *Let $\mathcal{H} = \otimes^k \mathbb{C}^2 \otimes H_{k+1} \otimes \cdots \otimes H_r$ be as above. Then an unentangled frame function for \mathcal{H} is the restriction of one for $\otimes^k \mathbb{C}^2 \otimes H$ with H the completion of $H_{k+1} \otimes \cdots \otimes H_r$.*

In light of this lemma we need only classify the unentangled frame functions on Hilbert spaces of the form $\otimes^k \mathbb{C}^2 \otimes H$ with H a separable Hilbert space of whose dimension is not 2. We note that if $\dim H = 1$ then a self-adjoint operator on H is a real scalar. Let $\mathcal{T}(H)$ be the space of trace class, self adjoint operators on H .

Theorem 5.2. *Let for each $J \subset \{1, \dots, k\}$, $\phi_J : F^{k-|J|} \rightarrow \mathcal{T}(H)$ (here as usual, we set $F^0 = \{\omega\}$). Let for $z \in F^k$ and let u be a state in H*

$$f(\sigma_J(z) \otimes u) = \sum_{L \subset J} (-1)^{|L-J|} \langle u | \phi_L(\tau_L(z)) | u \rangle. \quad (*)$$

Let

$$c = \text{tr} \phi_{\Omega_k}(\omega).$$

If $\{w_j\}$ is a UOB of $\otimes^k \mathbb{C}^2 \otimes H$ then

$$\sum f(w_j) = c.$$

If f is an unentangled frame function on $\otimes^k \mathbb{C}^2 \otimes H$ and if $J \subset \Omega_k$ we set for $z \in F^k$ and u a state in H

$$\gamma_J(u) = \sum_{L \subset J} f(\sigma_L(z) \otimes u),$$

then $\gamma_J(u) = \langle u | \phi_J(\tau_J(z)) | u \rangle$ with $\phi_J : F^{k-|J|} \rightarrow \mathcal{T}(H)$ and f is given by $(*)$.

Proof. If f is an unentangled frame function on $\otimes^k \mathbb{C}^2 \otimes H$ and if z is a state in $\otimes^k \mathbb{C}^2$ then, as above, $f_z(u) = f(z \otimes u)$ is a frame function on H . Thus there exists a trace class self adjoint operator on H , $\alpha(z)$, such that $f_z(u) = \langle u | \alpha(z) | u \rangle$ for u a state in H . Similarly we see that for fixed u a state in H ,

$$z \longmapsto \langle u | \alpha(z) | u \rangle$$

defines an unentangled frame function on $\otimes^k \mathbb{C}^2$. Thus there exist for each u a state in H and for $J \subset \Omega_k$ we have a function $\xi_{u,J}$ defined by

$$\xi_{u,J}(\tau_J z) = \sum_{L \subset J} \langle u | \alpha(\sigma_L z) | u \rangle$$

for z a state in $\otimes^k \mathbb{C}^2$. Now for each fixed z a state in $\otimes^k \mathbb{C}^2$ and $J \subset \Omega_k$

$$u \longmapsto \xi_{u,J}(\tau_J z)$$

is a frame function on H . Thus

$$\xi_{u,J}(\tau_J z) = \langle u | \phi_J(\tau_J z) | u \rangle$$

and the rest of the argument is clear. We also note that we can run this argument backwards using the result with $H = \mathbb{C}$ to prove the converse (second part of the theorem). \square

Remark 5.3. In the above, the frame functions can sum up to an arbitrary constant c on a UOB. To be a generalized version of a mixed state in quantum mechanics, as required by eq. (2), we need to set $c = 1$. Further, the definition in eq. (2.1) imposes the obvious inequalities on the sums in (*) that they be non-negative.

6. CONCLUSION

We have classified the unentangled frame functions, first for the multi-qubit system and then generally when tensor factors are of different dimensions, including separable Hilbert spaces. The proofs use the theory of Möbius functions to explicitly show the combinatorial nature of the multi-qubit UOB encountered in [4], and the multi-qubit unentangled frame functions quantify the result of measurements via such UOB. The structure of the frame functions thus revealed is sufficiently elegant that we surmise it points to interesting physical interpretations within the fundamentals of quantum mechanics. Indeed, qubit is the most basic quantum system, and it is common to see it as a subsystem in quantum algorithms, and it is also not unusual to use spatio-temporally separated measurements, which are by very definition, unentangled. Thus the information gleaned by such measurements falls within the context of analysis in this paper. On the other hand, if all the systems being measured have dimensions at least 3, then the conclusion of the unentangled Gleason's [3] theorem applies, which agree with the original Gleason's theorem. Another area where we often encounter a mix of systems is the Hydrogen atom. Its phase space is a spin bundle of dimension 2 tensored with $\ell^2(\mathbb{R}^3)$, so it is precisely a sub-case of the general case we discuss in Section 5. These are two of the more obvious examples, but underline a need to understand the significance of unentangled measurements.

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